Indian Statistical Institute, Bangalore Centre
B.Math. (III Year)/ M.Math. (II year) : 2014-2015
Semester I : Backpaper Examination
Markov Chains

09.01.2015 Time: 3 hours Maximum Marks : 100

Note: State clearly the results you are using in your answers.

1. (20 marks) $\{X_n : n = 0, 1, 2, \cdots\}$ is a Markov chain with state space $S = \{1, 2, 3, 4, 5, 6, 7\}$ with transition probability matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

Find the set of transient states, and the irreducible closed set(s) of recurrent states.

2. (20 marks) Consider the discrete time queueing Markov chain $\{X_n : n = 0, 1, \cdots\}$, with a_k = Prob. (k customers arrive during a service period), $k = 0, 1, 2, \cdots$ During each service period exactly one customer is served if anyone is waiting; no service given if no one is waiting. Let X_n denote the number of people waiting in the queue at the end of the $n-$th service period.

   (i) If either $a_0 = 0$, or $a_0 + a_1 = 1$, show that the Markov chain is not irreducible.

   (ii) If $a_0 > 0$ and $a_0 + a_1 < 1$, show that the Markov chain is irreducible.

3. ($15 + 5 = 20$ marks) (i) Consider the Ehrenfest urn model $\{X_n\}$ with $2d$ balls. Show that the binomial distribution with parameters $2d$ and $\frac{1}{2}$ is the unique stationary probability distribution for $\{X_n\}$.

   (ii) Show that $\{X_n\}$ is time reversible.

4. (10+10 = 20 marks) Let $N(\cdot)$ be a time homogeneous Poisson process with arrival rate $\lambda > 0$. For $n = 1, 2, \cdots$ let $W_n = \inf\{t \geq 0 : N(t) = n\}$ denote the time of n-th arrival.

   (i) Show that $P(W_n < \infty) = 1$ for any $n \geq 1$.

   (ii) Show that the joint probability density function of W_1, W_2, W_3 is given by

$$f(y_1, y_2, y_3) = \begin{cases} \lambda^3 e^{-\lambda y_3}, & 0 < y_1 < y_2 < y_3 < \infty \\ 0, & \text{otherwise} \end{cases}$$

5. (20 marks) Show that an infinite server $M/M/\infty$ queue can be regarded as a birth and death process, and find its infinitesimal birth and death rates.